§ 8. Building Data – Part 2
(Regular representation of the pushforw. of the struct. sheaf of a Abelian cover)

Let $\pi: X \to Y$ be a Galois cover with group G.
Let us consider the pushforward of the structure sheaf of X, $\pi_* \mathcal{O}_X$. The action of G on X induces an action on $\pi_*\mathcal{O}_X$, which decomposes in eigensheaves V_χ, $\chi \in \mathrm{Irr}(G)$. More precisely, given an open set $\mathcal{U} \subseteq Y$, then

$$V_\chi(\mathcal{U}) := \left\{ \pi_\chi(f) = \frac{\chi(1_G)}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot g^* f \;\middle|\; f \in \mathcal{O}_X(\pi^{-1}(\mathcal{U})) \right\}$$

From representation theory, we naturally have $\pi_*\mathcal{O}_X(\mathcal{U}) = \bigoplus_{\chi \in \mathrm{Irr}(G)} V_\chi(\mathcal{U})$, so

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in \mathrm{Irr}(G)} V_\chi$$

Theorem V_χ are locally free sheaves of Y of rank $\chi^2(1_G)$. In particular, the representation of G on $\pi_*\mathcal{O}_X$ is the regular representation.

Proof: We prove the theorem only for G abelian group. Thus, we are going to prove V_χ are invertible sheaves of Y.
Let $q \notin \mathrm{supp}(D)$, and let V be a fundamental neigh. of q, namely $\pi^{-1}(V) = \bigsqcup_{g \in G} g \cdot \mathcal{U}$, $\mathcal{U}$ open set of X, and

$\pi|_{\mathcal{U}} : \mathcal{U} \to V$ is an isomorphism. We consider the function $\mathbb{1}_{\mathcal{U}} = \begin{cases} 1 & \text{if } p \in \mathcal{U} \\ 0 & \text{otherwise} \end{cases}$, and we project the function on the isotypic component of character χ, V_χ:

$$\tau_V^\chi := \sum_{g \in G} \overline{\chi(g)} \cdot \underbrace{g^* \mathbb{1}_{\mathcal{U}}}_{\parallel} \in V_\chi(V)$$

$$g^*\mathbb{1}_{\mathcal{U}}(p) = \mathbb{1}_{\mathcal{U}}(g^{-1} p) = \begin{cases} 1 & \text{if } g^{-1}p \in \mathcal{U} \iff p \in g\mathcal{U} \\ 0 & \text{otherwise} \end{cases} = \mathbb{1}_{g \cdot \mathcal{U}}$$

Let $f \in V_\chi(V)$, so $g^* f = \chi(g) f$ (since any irreducible character of an abelian group is 1-dimensional)
Then $f = \frac{f}{\tau_V^\chi} \cdot \tau_V^\chi$ on $\pi^{-1}(V)$.

↳ it is well-def invariant fct. on $\pi^{-1}(V)$ because $\forall g p \in g\mathcal{U}$, then $\mathbb{1}_{\mathcal{U}}^\chi(g \cdot p) = \frac{1}{|G|} \cdot \overline{\chi}(g) \cdot 1 \neq 0$.

Thus, we have

$$\begin{aligned} \mathcal{O}_{Y|_V}(V') &\xrightarrow{\sim} V_{\chi|_V}(V') \\ \alpha &\longmapsto \alpha \cdot \tau_V^\chi|_{V'} \end{aligned}$$

so $V_{\chi|_V} \cong \mathcal{O}_{Y|_V}$.

Let us consider $q \in \mathrm{supp}(D_h) \setminus \mathrm{Sing}(D)$; then $\pi^{-1}(p) \overset{1:1}{\longleftrightarrow} \frac{G}{\mathrm{Stab}_G(p)} = \frac{G}{\langle h \rangle}$ and we can construct an open neigh. V of q such that there is a fundam. neigh $\mathcal{U}$ for which $\pi^{-1}(V) = \bigsqcup_{g \in G/\langle h \rangle} g \cdot \mathcal{U}$, while $h \cdot \mathcal{U} = \mathcal{U}$, and $\tilde{\pi} : \mathcal{U}/_{\langle h \rangle} \to V$ is <u>iso</u>.

Furthermore, $\mathcal{U}$ can be chosen such that, given a point $p \in \mathcal{U}$ over q with $p \in T$, T irred. component of R_h, then $\mathcal{U}$ has coordinates $(t, z_2, \dots z_n)$ with $T = (t=0)$, and $h : \mathcal{U} \to \mathcal{U}$ acts as $(t, z_2, \dots z_n) \xrightarrow{h} (\underset{\substack{\| \\ e^{\frac{2\pi i}{|h|}}}}{\xi} t, z_2, \dots z_n)$.

Def Given $\chi \in \mathrm{Irr}(G)$ and $g \in G$, we define $0 \leq r_\chi^g \leq |g|-1$ as the unique integer for which

$$\chi(g) = \xi^{r_\chi^g}, \qquad \xi = e^{\frac{2\pi i}{|g|} \cdot 1}$$

Let us fix the function $t^{r_\chi^h} \mathbb{1}_{\mathcal{U}}$ and let us take the invariant function of character χ

$$\tau_V^\chi := \sum_{g \in G/\langle h \rangle} \overline{\chi(g)}\, g^* \underbrace{\left(t^{r_\chi^h} \mathbb{1}_{\mathcal{U}}\right)} \in V_\chi(V)$$

$$g^*\left(t^{r_\chi^h} \mathbb{1}_{\mathcal{U}}\right)(p) = \left(t^{r_\chi^h} \mathbb{1}_{\mathcal{U}}\right)(g^{-1} p) = \begin{cases} t^{r_\chi^h} & \text{if } g^{-1} p \in \mathcal{U} \iff p \in g\mathcal{U} \\ 0 & \text{otherwise} \end{cases}$$

$$= t(g^{-1}p)^{r_\chi^h} \mathbb{1}_{g \cdot \mathcal{U}} = (g^* t)^{r_\chi^h} \mathbb{1}_{g \cdot \mathcal{U}}$$

Let us consider $f \in V_\chi(V)$, so $f \in \mathcal{O}_X(\pi^{-1}(V))$ is an invariant function of character χ.

$f|_{\mathcal{U}}$ can be written as $f = \sum a_m(z_2, \dots z_n) \cdot t^m$

$$\Rightarrow \quad h\cdot f = \chi(h)\cdot f = \sum a_m(z_2,\dots z_n)\cdot \xi^m \cdot t^m$$

$$\chi(h) = \xi^{r_\chi^h}, \qquad \xi^{r_\chi^h}\cdot f = \sum a_m(z_2,\dots z_n)\cdot \xi^{r_\chi^h} t^m$$

$$\Rightarrow \left(a_m \neq 0 \iff \xi^m = \xi^{r_\chi^h} \iff m = r_\chi^h + \alpha|h|\right)$$

$$\Rightarrow \quad f_{|\mathcal{U}} = \sum a_m(z_2,\dots z_n)\cdot t^m = \sum a_m t^{r_\chi^h + \alpha|h|} =$$

$$= t^{r_\chi^h}\Big(\underbrace{\sum a_m \left(t^{|h|}\right)^\alpha}_{= s(t^{|h|})}\Big) = t^{r_\chi^h} s(t^{|h|})$$

Instead, $x \in \mathcal{U}$, $\quad f(gx) = (f\circ g)(x) = (g^{-1})^* f(x) = \overline{\chi(g)}\, f(x)$

$$= \bar\chi(g)\cdot t_{(x)}^{r_\chi^h} s\left(t^{|h|}_{(x)}\right) = \bar\chi(g)(t\circ g^{-1})^{r_\chi^h}_{(gx)}\cdot s\left((t\circ g^{-1})^{|h|}_{(gx)}\right)$$

$$\Rightarrow \quad f_{|g\mathcal{U}} = \bar\chi(g)\,(t\circ g^{-1})^{r_\chi^h} s\left((t\circ g^{-1})^{|h|}\right)_{|g\mathcal{U}}$$

$$= \bar\chi(g)\,(g^* t)^{r_\chi^h}\cdot s\left((g^* t)^{|h|}\right)$$

$$\Rightarrow \quad f\cdot \mathbb{1}_{g\mathcal{U}} = \bar\chi(g)(g^* t)^{r_\chi^h}\cdot \mathbb{1}_{g\mathcal{U}}\, s\left((g^* t)^{|h|}\right)$$

$$\Rightarrow \quad f = \sum_{g\in G/\langle h\rangle} f\, \mathbb{1}_{g\mathcal{U}} = \tau_V^\chi \cdot \underbrace{\sum s\left((g^* t)^{|h|}\right)\mathbb{1}_{g\mathcal{U}}}_{\sum_{g\in G/\langle h\rangle} g^*\left(s(t^{|h|})\mathbb{1}_{\mathcal{U}}\right)}$$

invariant G-function $\in \mathcal{O}_y(V)$

$$\Rightarrow \quad f = \alpha\cdot \tau_V^\chi \quad \text{with } \alpha \in \mathcal{O}_y(V)$$

Furthermore, the same argument works for any $V' \subseteq V$ as $f \in V_\chi(V')$ is a fuction of $\mathcal{O}_X(\pi^{-1}(V'))$ and can be written as $f = \sum a_m t^m$. Thus

$$\mathcal{O}_{Y|_V}(V') \xrightarrow{\sim} V_{\chi}|_V(V')$$
$$\alpha \longmapsto \alpha \cdot T$$

Thus, V_χ is locally free of rank 1 out of a codimension 2 locus $Y \setminus \mathrm{Sing}(D)$

+ it is torsion free (because V_χ is a subsheaf of $\pi_* \mathcal{O}_X$, which is torsion free by the fact that X is (and reduced) normal, Y is smooth, and $\pi: X \to Y$ is dominant)

$\Rightarrow$ V_χ is a locally free sheaf of rank 1.

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From now on, G is an abelian group.

Let us denote $L_\chi := V_\chi^{-1}$. We have the decomposition

$$\boxed{\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} L_\chi^{-1}}$$

Def The set of divisors $\{D_g\}_{g \in G}$ and line budles $\{L_\chi\}_\chi$ are called <u>Building Data</u> of $\pi: X \to Y$.

We notice that the cocycles of V_χ are:

$$\begin{array}{ccccc} \mathcal{O}_Y(V_1\cap V_2) & \longrightarrow & V_\chi(V_1\cap V_2) & \longrightarrow & \mathcal{O}_Y(V_1\cap V_2) \\ \alpha & \longmapsto & \alpha\,\tau_V^\chi = \alpha\,\frac{\tau_{V_1}^\chi}{\tau_{V_2}^\chi}\,\tau_{V_1}^\chi & \longmapsto & \frac{\tau_{V_1}^\chi}{\tau_{V_2}^\chi}\cdot\alpha \end{array}$$

$$\Rightarrow \quad g_{V_2V_1} = \frac{\tau_{V_1}^\chi}{\tau_{V_2}^\chi} \in \mathcal{O}_Y(V_1\cap V_2)$$

Then the cocycles of $L_\chi = V_\chi^{-1}$ are $g_{V_2V_1} = \dfrac{\tau_{V_2}^\chi}{\tau_{V_1}^\chi}$

$\Rightarrow$ a global (meromorphic) section of $\pi^* L_\chi$ is

$$\boxed{s_\chi := \{(\pi^{-1}(V), \tau_V^\chi)\}_{V\subseteq Y}}$$

This is actually holom. as s_χ is holom. out of a codim. 2 locus (s_χ is on $Y\setminus\mathrm{Sing}(D)$) So s_χ is holom. on Y by Hartogs Theorem.

Indeed, $\tau_{V_2}^\chi = \dfrac{\tau_{V_2}^\chi}{\tau_{V_1}^\chi}\cdot\tau_{V_1}^\chi$ on $\pi^{-1}(V_1\cap V_2)$

We can finally state and prove Pardini Existence Theorem.

Pardini Existence Theorem

Let Y be a smooth complete algebraic variety and let $\pi : X \to Y$ be an abelian cover of Y with Galois group G and X normal.
Then, for any pairs of characters $\chi, \eta \in G^*$

$$(*) \qquad L_\chi + L_\eta = L_{\chi\eta} + \sum_{g \in G} \left\lfloor \frac{r_\chi^g + r_\eta^g}{|g|} \right\rfloor D_g$$

Conversely, given

- a collection of line bundles $\{\mathcal{L}_\chi\}_{\chi \in G^*}$ of Y labeled by the characters of G;
- a collection of effective DIVISORS $\{D_g\}_{g \in G}$ indexed by the elements of G;

with the property that $D := \sum_{g \in G} D_g$ is reduced and the linear equations $(*)$ hold for any pair $\chi, \eta \in G^*$, then it there exists a unique abelian cover (up to isomorphisms of G-covers) $\pi : X \to Y$ with Galois group G and X normal, whose building data are $\{\mathcal{L}_\chi\}_{\chi \in G^*}$ and $\{D_g\}_{g \in G}$.